Lecture 4 - Sep 16

Math Review

Implication: Alternative Exp. in English
Logical Quantifiers: Proof Strategies
Sets: Enumerations vs. Comprehension

Announcements/Reminders

- Today's class: notes template posted
- Event-B Summary Document
- Priorities:
  + Lab1 → Due: This Tuesday (Sep 16)
  + Lab2 → Due: Next Tuesday (Sep 23)

given p (ante.) is true, the only way to make $p \Rightarrow q$ true is when q is also true.

$\boxed{p \Rightarrow q}$

$p \Rightarrow q$ — p: sufficient, q: necessary

p: snow storm
q: cancel class

Expressing Implications

$\boxed{q \text{ if } p}$, p is sufficient for q

$\boxed{p \Rightarrow \text{True} \equiv \text{True}}$

(c1) q unless ¬p (c2)

p	q	$p \Rightarrow q$
true	true	true
true	false	false
false	true	true
false	false	true

if p

if ¬p, then don't care what q is

neither c1 ¬q

nor c2 ¬(¬p) "p"

p	q	$p \Rightarrow q$
true	true	true
true	false	false
false	true	true
false	false	true

c1: q

c2: ¬p

p only if q, q is necessary for p

p	q	$p \Rightarrow q$
true	true	true
true	false	false
false	true	true
false	false	true

p.

¬q

¬p ↳ don't care about q.

e.g. Formulate "$x > 0$ (q) if $y \leq 10$ (p)"

↳ $y \leq 10 \Rightarrow x > 0$

$p \Leftrightarrow q$ (if; only if)

1. p if q ($p \Leftarrow q$, $q \Rightarrow p$)
2. p only if q ($p \Rightarrow q$)

$\mathbb{Z}$ set of integers

$-\infty, \cdots, -1, 0, 1, \cdots, +\infty$

$\mathbb{N}$ set of natural numbers

$0, 1, 2, \cdots, +\infty$

$\mathbb{N}_1$ $1, 2, \cdots, +\infty$ (positive integers).

3 numbers $\boxed{1..3}$

8 numbers $\boxed{4..11}$

$$\forall i, j \cdot i \in \boxed{\mathbb{N}} \wedge j \in \boxed{\mathbb{Z}} \Rightarrow \underline{P(i, j)}$$

need to consider

$3 * 8 = 24$

combinations

of i, j.

need to consider

<u>all</u> combinations

of (i, j)

Logical Quantifiers: Examples

$\forall i \bullet \boxed{i \in \mathbb{N}} \Rightarrow i \geq 0$ T

$i \in \{0, 1, 2, \ldots\}$

F $\forall i \bullet \underline{i \in \mathbb{Z} \Rightarrow i \geq 0}$ witness: -1 $\boxed{-1 \in \mathbb{Z}} \Rightarrow \boxed{-1 \geq 0}$ T F F

false $\forall i, j \bullet \underline{i \in \mathbb{Z} \wedge j \in \mathbb{Z} \Rightarrow i < j \vee i > j}$

witness: choose i, j s.t. $i = j$.

T $\exists i \bullet i \in \mathbb{N} \wedge i \geq 0$ witness: 0

$0 \in \mathbb{N} \wedge 0 \geq 0 \equiv$ T.

T $\exists i \bullet i \in \mathbb{Z} \wedge i \geq 0$ witness: 0

T $\exists i, j \bullet i \in \mathbb{Z} \wedge j \in \mathbb{Z} \wedge (i < j \vee i > j)$

witness: $i = 2$, $j = 3$.

Logical Quantifiers: Examples

$R(i) \triangleq$ student i enrolled in 3342
$P(i) \triangleq$ student i received A+

$\triangleq$ is defined as.

Goal: show $R(i) \Rightarrow P(i) \equiv$ True.

How to $\boxed{\text{prove}}$ $\forall i \bullet R(i) \Rightarrow P(i)$?

trivial ← (1) show $\neg R(i)$ ∵ zero of $\Rightarrow$: false $\Rightarrow P \equiv$ true

harder ← (2) show $R(i)$, $P(i)$ (every i sat R also sat P)

How to $\boxed{\text{prove}}$ $\exists i \bullet R(i) \wedge P(i)$?

Goal: show $R(i) \wedge P(i) \equiv$ True.

(1) give a witness i that sat both R and P.

Goal: show $R(i) \Rightarrow P(i) \equiv$ False.

How to $\boxed{\text{disprove}}$ $\forall i \bullet \underbrace{R(i)}_{T} \Rightarrow \underbrace{P(i)}_{F}$?

(1) give a witness j s.t. $R(j)$ but $\neg P(j)$.

How to $\boxed{\text{disprove}}$ $\exists i \bullet R(i) \wedge P(i)$? harder.

Goal: show $R(i) \wedge P(i) \equiv$ False.

(1) show $\neg R(i)$: false $\wedge P \equiv$ false

(2) $R(i)$, $\neg P(i)$ — every i sat R does not sat P.

Prove/Disprove Logical Quantifications

- Prove or disprove: $\forall x \bullet (x \in \mathbb{Z} \land 1 \leq x \leq 10) \Rightarrow x > 0.$

$x \in 1..10$ → each member in this interval is > 0.

($R(x)$ is not false)

- Prove or disprove: $\forall x \bullet (x \in \mathbb{Z} \land 1 \leq x \leq 10) \Rightarrow x > 1.$

Exercise

- Prove or disprove: $\exists x \bullet (x \in \mathbb{Z} \land 1 \leq x \leq 10) \land x > 1.$

$x \in 1..10$

witness: 2

Disprove using witness $x = 1$

$\boxed{1 \in \mathbb{Z} \land 1 \leq 1 \leq 10 \land 1 > 1}$ F

- Prove or disprove that $\exists x \bullet (x \in \mathbb{Z} \land 1 \leq x \leq 10) \land x > 10?$

Exercise

wrong statements: to disprove $\exists$, not sufficient with a single witness.

Sets: Definitions and Membership

No ordering: $\underbrace{\{1, 2, 3\}}_{\text{set enumeration (members are explicit)}} = \{2, 3, 1\}$

No duplicates: $\{1, \underline{\underline{2}}, 3, \underline{\underline{2}}\}$ ✗

① $\{\underline{x} \mid x \in \mathbb{N} \land 0 \le x \le 2\} = \{0, 1, 2\}$

② $\{2 \cdot x \mid x \in \mathbb{N} \land 0 \le x \le 2\} = \{0, 2, 4\}$

Set Comprehension

$$\{\ \underline{\text{expression: form of member in the final set}}\ \mid\ \underline{\boxed{\text{Constraint}}\text{: values }\underline{\underline{\text{sat}}}\text{ this can be included in the set}}\ \}$$

implicit about what set members are

$v1, v2, v3$

③ $\{(x, y) \mid x \in \mathbb{N} \land y \in \mathbb{N} \land 1 \le x \le 2 \land 3 \le y \le 4\}$

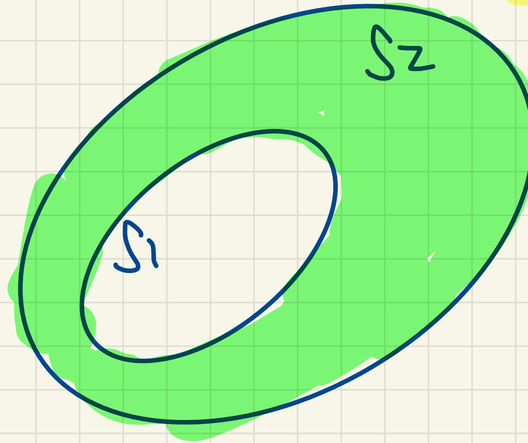
$= \{(1,3), (1,4), (2,3), (2,4)\}$

Q. How many sets of size 3 can you make out of 1, 2, 3, 4, 5

Relating Sets

subset

$S_1 \subseteq S_2$



$S_2 \setminus S_1 = 0$

$\Rightarrow S_1 = S_2$

set diff

$\left| \underbrace{S_2 \setminus S_1}_{S} \right| > 0$

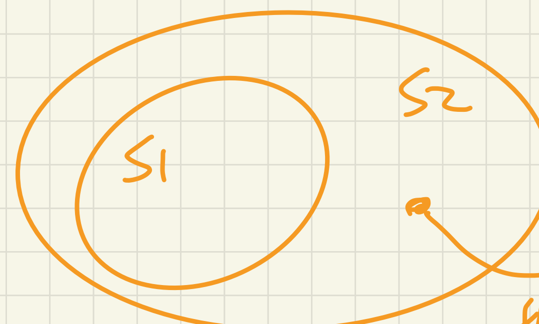
all members in S_2 but not in S_1

proper subset

$S_1 \subset S_2$

$\iff S_1 \subseteq S_2 \land S_1 \neq S_2$

$\iff S_1 \subseteq S_2 \land (\exists x . x \in S_2 \land x \notin S)$



witness that is $\in S_2$ & $\notin S_1$